2. (b) (i) Let, $\{G_i : i \in I\}$ be any collection of open sets in a metric space (X, ρ). Prove that $\bigcup_{i\in I} G_i$ is open in X. (3)

Proof : Suppose, $\{G_i\}_{i\in I}$ (where I = denotes the indexing set) be a family of open sets.

Let, $\xi \in \bigcup_{i\in I} G_i$. Clearly $\xi \in G_\alpha$ for some $i\alpha \in I$

Since, each G_i is open, we have ξ is an interior point of G_i

So, $\exists$ an open ball, say $S_r(\xi)$ such that $S_r(\xi) \subset G_i$

$\therefore$ $S_r(\xi) \subset \bigcup_{i\in I} G_i$

So, ξ is an interior point of $\bigcup_{i\in I} G_i$

Hence, $\bigcup_{i\in I} G_i$ is an open set.

(ii) Let, (X, ρ) and (Y, d) be a metric spaces and let $f : X \to Y$ be a function. If f is continuous then show that for every open set $V \subset Y$, the set $f^{-1}(V)$ is open in X. (3)

Proof: Suppose, f is continuous function from X to Y.

If V is an open set $(\neq \emptyset)$ in Y and $G = f^{-1}(V)$, then

G is open if $G = f^{-1}(V)$ is empty.

So let, $G \neq \emptyset$

Take, $u \in G$, $f(u) \in V$.

Since, V is open, $f(u)$ is an interior point of V. and let,

$B_\varepsilon(f(u)) \subseteq V$.

Using continuity of f at u, So corresponding to this $\varepsilon > 0$, we find a '+ve' δ such that $f(B_\delta(u)) \subseteq B_\varepsilon(f(u)) \subseteq V$.

i.e. $B_\delta(u) \subseteq f^{-1}(V) = G$.

i.e. $B_\delta(u) \subseteq G$.

i.e. u is an interior point of G.

Since, u being any elements of G, so G is open in X.

i.e. $f^{-1}(V)$ is open in X.

2.c.(i) If ρ is a metric on a set X and if σ is defined by $\sigma(x,y) = \frac{\rho(x,y)}{1+\rho(x,y)}$, $x,y \in X$, then show that σ is also a metric on X.

Ans: For $x,y \in X$, $\sigma(x,y) = \frac{\rho(x,y)}{1+\rho(x,y)} \geq 0$

$= 0$ iff $x = y$, $\because \rho(x,y)$ is a metric.

Now, $\sigma(x,y) = \sigma(y,x)$ is trivial.

For $x,y,z \in X$

$$\sigma(x,y) + \sigma(y,z) = \frac{\rho(x,y)}{1+\rho(x,y)} + \frac{\rho(y,z)}{1+\rho(y,z)}$$

$$\geq \frac{\rho(x,y)}{1+\rho(x,y)+\rho(y,z)} + \frac{\rho(y,z)}{1+\rho(y,z)+\rho(x,z)}$$

$$= \frac{\rho(x,y)+\rho(y,z)}{1+\rho(x,y)+\rho(y,z)}$$

$$= \frac{1}{1+\frac{1}{\rho(x,y)+\rho(y,z)}}$$

$$\geq \frac{1}{1+\frac{1}{\rho(x,z)}} \quad [\because \rho(x,z) \leq \rho(x,y)+\rho(y,z)]$$

$$= \frac{\rho(x,z)}{1+\rho(x,z)}$$

$$= \sigma(x,z).$$

i.e. $\sigma(x,y) + \sigma(y,z) \geq \sigma(x,z)$.

i.e. $\sigma(x,z) \leq \sigma(x,y) + \sigma(y,z)$.

$\therefore$ Triangle inequality holds.

Hence, σ is also a metric on X.

(ii) Let, (X,ρ) be a metric space and let $A \subset X$. Then show that $\bar{A} = \{x : x \in X ; \rho(x,A) = 0\}$, where $\bar{A}$ is the closure of A w.r.to the metric ρ and $\rho(x,A)$ denotes the distance of x from A. (3)

Ans: Let, $x \in \bar{A}$ and $\epsilon > 0$ be arbitrary. Then $\exists$ a $\in A \cap B(x,\epsilon)$, so that $\rho(x,A) \leq \rho(x,a) < \epsilon$, for arbitrary $\epsilon > 0$.

Hence, $\rho(x,A) = 0$.

If $x \notin \bar{A}$, then $\exists\, r > 0$ such that $A \cap B(x,r) = \phi$ i.e. $d(a,x) \geq r$, $\forall a \in A$ and hence $d(x,A) \geq r > 0$, showing that $x \notin \{y \in X : d(y,A) = 0\}$.

5. C. (i) If a sub-set E of a metric space (X,ρ) is connected then show that its closure $\bar{E}$ is also connected. Give an example to show that connectedness of $\bar{E}$ may not imply connectedness of E. (5+1=6)

Proof: Let, $\bar{E}$ be not connected, then $\bar{E} = B \cup C$, where B and C are non-empty disjoint closed sets in $\bar{E}$.

Now, $E \subset \bar{E} = B \cup C$ and as E is connected either $E \subseteq B$ or, $E \subseteq C$.

Thus, either $\bar{E} \subseteq \bar{B} = B$ or $\bar{E} \subseteq \bar{C} = C$.

which is a contradiction.

Hence, closure of a connected set is connected.

(ii) Let, (X,ρ) and (Y,d) be a metric spaces and let $f: X \to Y$ be continuous. Show that if X is connected then the range $f(X)$ is also connected. 6

[Ans: Page-2]

5.

d. (i) Let, (X, ρ) and (Y, d) be metric spaces and $f: X \to Y$ be a function. When is the function f is said to be uniformly continuous? Show that if X is compact then f is uniformly continuous. (1+6=7)

Ans: Let, $f: (X, \rho) \to (Y, d)$, f is said to be uniformly continuous if given $\epsilon > 0$ there is a '+ve' δ such that $d(f(x_1), f(x_2)) < \epsilon$ whenever $\rho(x_1, x_2) < \delta$.

■ Let, (X, ρ) be a compact metric space and (Y, d) be any metric space. Also let, $f: (X, \rho) \to (Y, d)$ be continuous function.

If $\epsilon > 0$, $x \in X$ by continuity of f at x, we find a '+ve' $\delta(x)$ such that $d(f(x'), f(x)) < \epsilon/2$, whenever $\rho(x', x) < \delta(x)$.

Now, consider the family of open balls $\{B(x, \frac{1}{2}\delta(x))\}_{x \in X}$.

clearly, this family $\{B(x, \frac{1}{2}\delta(x))\}_{x \in X}$ is an open cover for X. But X is compact, so there is a finite sub-cover $\{B(x_1, \frac{1}{2}\delta(x_1)), B(x_2, \frac{1}{2}\delta(x_2)), \ldots, B(x_n, \frac{1}{2}\delta(x_n))\}$

Let us choose δ such that $\delta < \min_{1 \le i \le n} (\frac{1}{2}\delta(x_i))$

Let, $u, v \in X$ with $\rho(u, v) < \delta$.

Let, $u \in B(x_i, \frac{1}{2}\delta(x_i))$, for some i $[1 \le i \le n]$.

Then $\rho(v, x_i) \le \rho(v,u) + \rho(u, x_i)$.

$$< \delta + \frac{1}{2}\delta(x_i)$$
$$< \frac{1}{2}\delta(x_i) + \frac{1}{2}\delta(x_i)$$
$$< \frac{1}{2}\delta(x_i).$$

$\therefore$ $d(f(u), f(v)) \le d(f(u), f(x_i)) + d(f(x_i), f(v))$

$$< \varepsilon/_2 + \varepsilon/_2 = \varepsilon.$$

Hence, the theorem.

5. d. (ii) For any two real numbers x, y define $\sigma(x,y) = \left|\frac{|x|}{1+|x|} - \frac{|y|}{1+|y|}\right|$. Show that σ is a metric on the set of real numbers. (5)

Ans: For any $x, y \in \mathbb{R}$, $\sigma(x,y) = \left|\frac{|x|}{1+|x|} - \frac{|y|}{1+|y|}\right| \geqslant 0$

$= 0$ iff $x = y$.

For any $x, y \in \mathbb{R}$, $\sigma(x,y) = \sigma(y,x)$ is trivial.

If for $x, y, z \in \mathbb{R}$,

$$\sigma(x,y) + \sigma(y,z) = \left|\frac{|x|}{1+|x|} - \frac{|y|}{1+|y|}\right| + \left|\frac{|y|}{1+|y|} - \frac{|z|}{1+|z|}\right|$$

$$\geqslant \left|\frac{|x|}{1+|x|} - \cancel{\frac{|y|}{1+|y|}} + \cancel{\frac{|y|}{1+|y|}} - \frac{|z|}{1+|z|}\right|.$$

$$= \left|\frac{|x|}{1+|x|} - \frac{|z|}{1+|z|}\right|$$

$$= \sigma(x,z).$$

i.e. $\sigma(x,z) \le \sigma(x,y) + \sigma(y,z)$.

Triangle inequality holds.

Hence, σ is a metric on the set of all real numbers.

5. c. (ii) Let, (X, ρ) and (Y, d) be metric spaces and let $f: X \to Y$ be continuous. Show that if X connected then the range $f(X)$ is also connected. (6).

Ans: Let, f is continuous map from a metric space (X, ρ) to another (Y, d).

Case-I If $A = \phi$, then $f(A) = f(\phi) = \phi$. Since, the null set is considered to be connected, there is nothing to prove.

case-II. If A is a singleton set, say $A=\{a\}$, then $f(A)=\{f(a)\}$ is also a singleton set. In this case also, $f(A)$ is connected.

case-III. Let, $A \subset X$ contains atleast two points. We are to prove that $f(A)$ is connected in (Y,d).

If possible let $f(A)$ is disconnected, then $\exists$ two non-empty sets G_1 and G_2 open in (Y,d) such that

$$f(A) \subset G_1 \cup G_2 \quad , f(A) \cap G_1 \neq \phi \quad , f(A) \cap G_2 \neq \phi$$

but $f(A) \cap (G_1 \cap G_2) = \phi$

Since, G_1, G_2 are open in the metric space (Y,d). and f is continuous, then their preimages $f^{-1}(G_1)$ and $f^{-1}(G_2)$ are open in (X,ρ).

Also, $A \cap (f^{-1}(G_1) \cap f^{-1}(G_2)) = f^{-1}(f(A)) \cap \{f^{-1}(G_1) \cap f^{-1}(G_2)\}$

$$= f^{-1}\{f(A) \cap G_1 \cap G_2\}.$$

$$= f^{-1}(\phi)$$

$$= \phi$$

Since, $f(A) \subset G_1 \cup G_2$, it follows that $A \subset f^{-1}(G_1 \cup G_2)$

$$= f^{-1}(G_1) \cup f^{-1}(G_2)$$

Finally, $G_1 \cap f(A) \neq \phi \Rightarrow f^{-1}(G_1 \cap f(A)) \neq \phi$

$$\Rightarrow f^{-1}(G_1) \cap A \neq \phi$$

Similarly, $f^{-1}(G_2) \cap A \neq \phi$

Thus, we can express $A = A_1 \cup A_2$, where

$A_1 = f^{-1}(G_1) \cap A$ and $A_2 = f^{-1}(G_2) \cap A$, $A_1 \cap A_2 = \phi$

consequently A is disconnected in (X,ρ). This is a contradiction to our hypothesis that A is connected.

$\therefore f(A)$ must be connected in the metric space (Y,d). This completes the proof.

5. b. (i) Let, (X, ρ) and (Y, σ) be metric spaces and let f and g be continuous function from X to Y. Show that the set $\{x : x \in X ;\ f(x) = g(x)\}$ is closed. Hence or otherwise prove that if $A \subset X$ is dense in X and $f(x) = g(x)$ for $x \in A$, then $f(x) = g(x)$, $\forall x \in X$. (4+3=7)

Proof: Let, $A = \{x \in X : f(x) = g(x)\}$. and $a \in X \setminus A$. Then $f(a) \neq g(a)$.

and hence $\sigma(f(a), g(a)) = \epsilon$ (say) > 0. Since f and g are continuous functions, $\exists\ \delta_1, \delta_2 > 0$ such that $\rho(x, a) < \delta_1$ $(x \in X) \Rightarrow \sigma(f(x), f(a)) < \epsilon/3$

and $\rho(x, a) < \delta_2\ (x \in X) \Rightarrow \sigma(g(x), g(a)) < \epsilon/3$.

Let, $\delta = \min(\delta_1, \delta_2)$..

Then $\delta > 0$ $\rho(x, a) < \delta\ (x \in X)$ ~~$\Rightarrow \sigma(f(x), f(a)) \leq$~~

$$\Rightarrow \sigma(f(a), g(a)) \leq \sigma(f(a), f(x)) + \sigma(f(x), g(x)) + \sigma(g(x), g(a))$$

$$< \epsilon/3 + \sigma(f(x), g(x)) + \epsilon/3.$$

$$= \frac{2\epsilon}{3} + \sigma(f(x), g(x)).$$

$\Rightarrow \sigma(f(x), g(x)) > \frac{\epsilon}{3}$ $\left[\because \sigma(f(a), g(a)) = \epsilon. \quad \therefore \epsilon < \frac{2\epsilon}{3} + \sigma(f(x), g(x)) \Rightarrow \frac{\epsilon}{3} < \sigma(f(x), g(x))\right].$

$\Rightarrow x \in X \setminus A$.

Hence, for each $a \in X \setminus A$, $\exists$ a $\delta > 0$ such that $B_\rho(a, \delta) \subseteq X \setminus A$. So, $X \setminus A$ is open and thus A is a closed set.

- From the above theorem we get the set $B = \{x \in X : f(x) = g(x)\}$ is closed in X, and by hypothesis $A \subseteq B$.

Then $X = \bar{A} \subseteq \bar{B} = B$, so that $B = X$.

Hence, $f(x) = g(x)$, $\forall x \in X$.

2.a.(i) Let, (X,ρ) be a metric space and let $\{x_n\}$ and $\{y_n\}$ be two sequences such that $x_n \to x$ and $y_n \to y$ as $n\to\infty$, where $x,y \in X$. Show that $\rho(x_n, y_n) \to \rho(x,y)$ as $n\to\infty$. (3)

Proof: Let, (X,ρ) be a metric space.

Since, $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$.

For given $\epsilon > 0$, $\exists$ two constant m, k $\in \mathbb{N}$ (~~[illegible]~~) such that $\rho(x_n, x) < \epsilon/2$, ~~whenever~~ $\forall\, n \geq m$.

and $\rho(y_n, y) < \epsilon/2$, $\forall\, n \geq k$.

Thus for $r = \max(m, k)$, $n \in \mathbb{N}$, $n \geq r$.

we have, $|\rho(x_n, y_n) - \rho(x,y)| \leq \rho(x_n, x) + \rho(y_n, y)$. [$\because \rho$ is a metric space]

$< \epsilon/2 + \epsilon/2 = \epsilon$, $n \geq r$

and hence, $\rho(x_n, y_n) \to \rho(x,y)$ as $n\to\infty$.

1.a. Let, X be a non-empty set. For any two points $x,y \in X$. Define

$$\rho(x,y) = \begin{cases} 0, & \text{if } x = y \\ 1, & \text{if } x \neq y. \end{cases}$$

Examine wheather ρ is a metric on X.

Ans: Clearly ρ is non-negetive and symmetric. From the def^n it follows that $x = y$ iff $\rho(x,y) = 0$.

Now, we are to prove the triangle inequality, i.e. $\forall\, x, y, z \in X$, $d(x,z) \leq d(x,y) + d(y,z)$.

If either $d(x,y) = 1$ or, $d(y,z) = 1$, then the inequality holds because the maximum possible values of $d(x,z)$ is 1, otherwise both $d(x,y) = 0$ and $d(y,z) = 0$. i.e. $x = y$ and $y = z$.

Hence, $x = z$. and consequently $d(x,z) = 0$. So, in all possible cases the triangle inequality satisfied.

This metric is termed as 'discrete metric' or 'trivial metric' on the set X.

1.b. Let $B[0,1]$ be the set of all bounded real valued functions on $[0,1]$, define $\rho(f,g) = \sup_{0\le x\le 1} |f(x)-g(x)|$, $f,g \in B[0,1]$. Examine wheather ρ is a metric on $B[0,1]$.

Ans: Clearly, $\rho(f,g) \geq 0$ always, further $f=g$, then $\rho(f,f)=0$

Conversely if $\rho(f,g)=0$, then $\sup_{0\le x\le 1} |f(x)-g(x)|=0$ i.e. $f(x)=g(x)$, $\forall x\in[0,1]$

i.e. $f\equiv g$.

Also, $\rho(f,g) = \rho(g,f)$, $\forall f,g \in c[a,b]$

$\therefore$ ρ is symmetric.

Finally if $f,g,h \in B[0,1]$, then we have

$|f(x)-h(x)| \le |f(x)-g(x)| + |g(x)-h(x)|$. $\forall x\in[0,1]$.

$\le \sup_{0\le x\le 1} |f(x)-g(x)| + \sup_{0\le x\le 1} |g(x)-h(x)|$.

$= \rho(f,g)+\rho(g,h)$.

$\sup_{0\le x\le 1} |f(x)-h(x)| = \rho(f,g)+\rho(g,h)$.

i.e. $\rho(f,h) = \rho(f,g)+\rho(g,h)$.

i.e. Triangle inequality is satisfied.

$\therefore$ ρ is a metric and $(B[0,1],\rho)$ is a metric space.

1. (d) Let, c be the set of all real valued continuous functions $[0,1]$. Define $\rho(f,g) = \int_0^1 |f(x)-g(x)|dx$, $f,g\in c$. Examine if ρ is a metric on c.

Ans: Clearly, $\rho(f,g)\geq 0$ always, further $f=g$ then $\rho(f,f)=0$.

Conversely if, $\rho(f,g)=0$ then $\int_0^1 |f(x)-g(x)|dx=0 \Rightarrow |f(x)-g(x)|=0$

$\Rightarrow f(x)=g(x)$

Also, $\rho(f,g)=\rho(g,f)$, $\forall f,g\in c[0,1]$ $\Rightarrow f\equiv g$.

$\therefore$ ρ is symmetric.

Finally if $f,g,h\in c[0,1]$, then we have

$|f(x)-h(x)| \le |f(x)-g(x)| + |g(x)-h(x)|$

$\le \int_0^1 |f(x)-g(x)|\,dx + \int_0^1 |g(x)-h(x)|\,dx$.

$= \rho(f,g) + \rho(g,h)$.

$\Rightarrow \int_0^1 |f(x)-h(x)|\,dx = \rho(f,g)+\rho(g,h)$.

$\Rightarrow \rho(f,h) \le \rho(f,g)+\rho(g,h)$.

$\because$ Triangle inequality holds. $\therefore \rho$ is a metric on $c[0,1]$.